A Pumping Lemma for Collapsible Pushdown Graphs of Level 2

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Collapsible Pushdown Systems (CPS)

- Higher-order pushdown systems (HOPS) [Maslov'76]
  - Pushdown systems with nested stack of . . . of stacks
  - Operation: push / pop for each stack level
- Motivation:

Theorem (Knapik, Niwinski, Urzyczyn '02)

trees of HOPS = trees of safe higher-order recursion schemes

Collapsible Pushdown Systems (CPS)

- Higher-order pushdown systems (HOPS) [Maslov'76]
  - Pushdown systems with nested stack of ... of stacks
  - Operation: push / pop for each stack level
- Collapsible pushdown system (CPS)
  Extension by "Collapse" operation
- defined by Hague, Murawski, Ong and Serre in '08
- Motivation:

Theorem (Knapik, Niwinski, Urzyczyn '02)

trees of HOPS = trees of safe higher-order recursion schemes

Theorem (Hague et al. '08)

trees of CPS = trees of higher-order recursion schemes

Basic Results on HOP Graphs / CP Graphs

Theorem (Carayol, Wöhrle '03)

HOPG/ε = Caucal-hierarchy

Corollary

MSO decidable on HOPG/ε

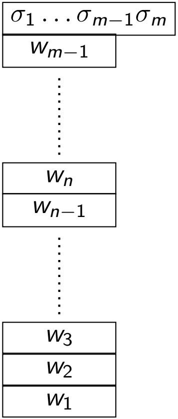
Theorem (Model checking on CPG/ε)

MSO	*undecidable*	*(Hague et al. '08)*
$L\mu$	*decidable*	*(Hague et al. '08)*
FO + Reach	*decidable on level 2*	*(Kartzow '10)*
FO	*undecidable on higher levels*	*(Broadbent)*

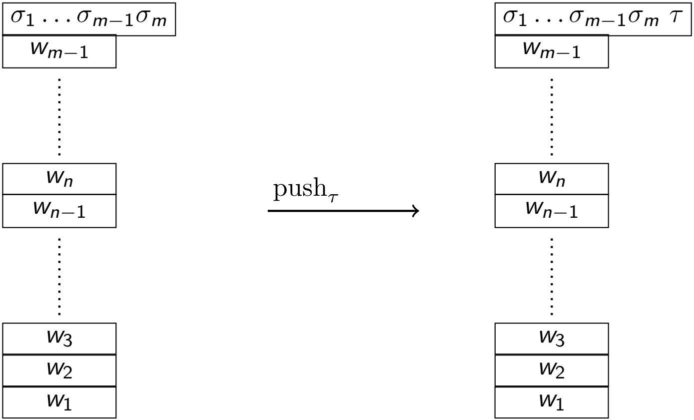
Structure of CPG

- We do *not* understand the structure of CPG
- No tools for "upper bounds":
  - $\mathfrak{G}$ is not a CPG?
  - $\mathfrak{G}$ is not a level i–CPG?
- Possible tool: pumping lemma for level i CPG
- Today: pumping lemma for level 2 CPG

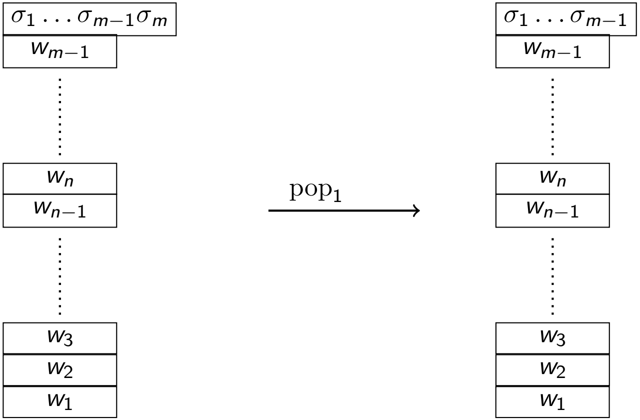
Stack Operations



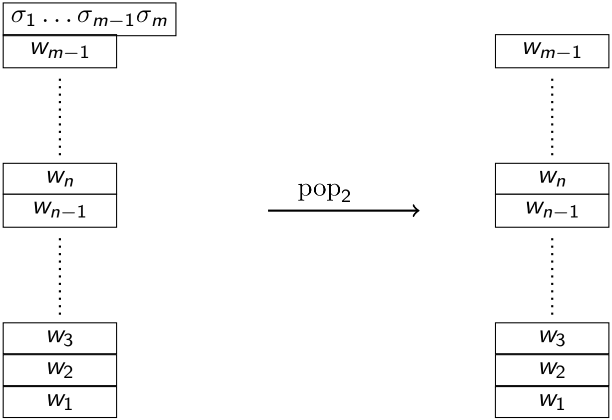
Stack Operations



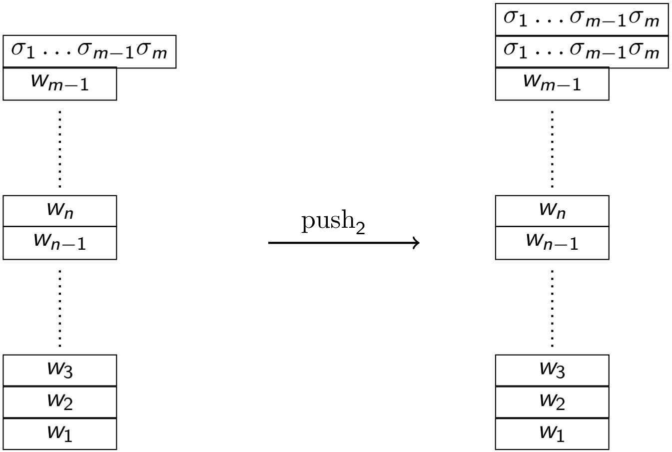
Stack Operations



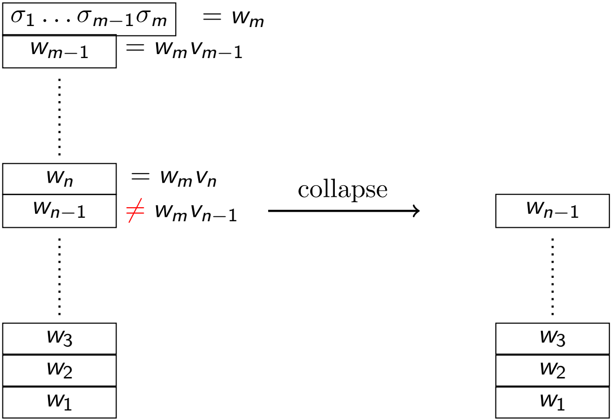
Stack Operations



Stack Operations

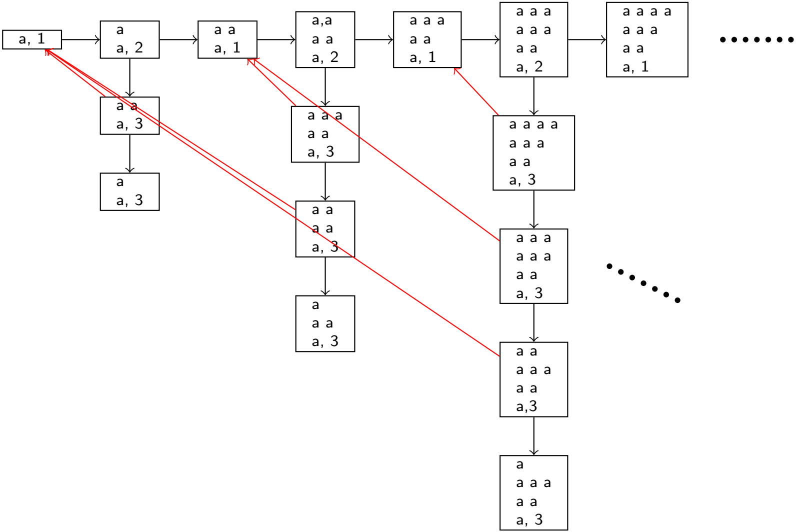


Stack Operations



Definition CPG

- Transition relation Δ:
  state + topmost letter $\mapsto$ new state + stack-operation
  e.g. $\delta = (q, \sigma) \mapsto (q', \mathrm{pop}_2)$
- Configuration (q, s) – q state, s stack (of level 2)
- $(q, s) \xrightarrow{\delta} (q', \mathrm{pop}_2(s))$
- CPG: configurations of CPS + labelled transition relation
- CPG$/\varepsilon$: ε-contraction of CPG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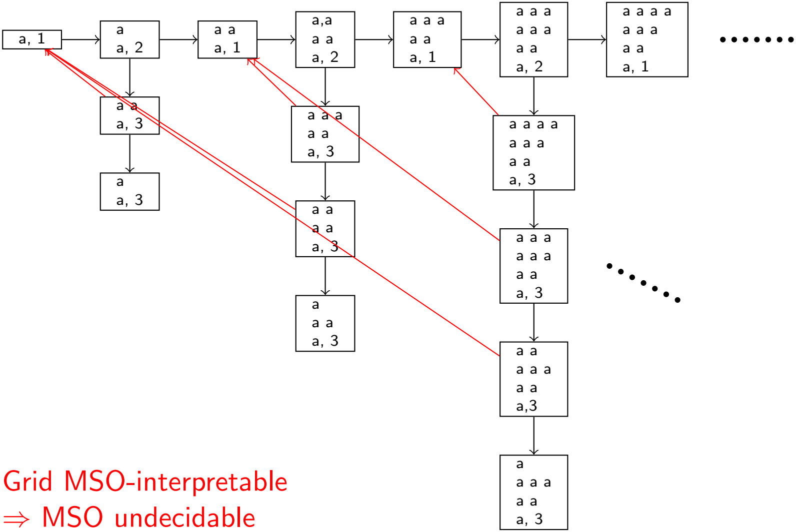
Example of CPG



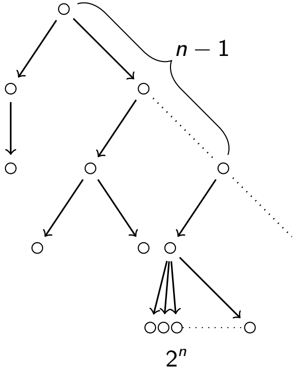
Grid MSO-interpretable

$\Rightarrow$ MSO undecidable

Another Example of CPG

Example

$\mathfrak{T} := (T, \mathrm{succ})$ with
$T := \{0\}^* \cup \{0^{n-1}1j : 0 \leq j \leq 2^n\}$
is a 2-CPG$/\varepsilon$



The Pumping Lemma for 2-CPG

Definition

$\mathfrak{G} = (V, (\xrightarrow{\gamma})_{\gamma\in\Gamma})$
$L \subseteq \Gamma^*$
$\xrightarrow{L} := \{(v_1, v_2) : v_1 \xrightarrow{\gamma_1} \dots \xrightarrow{\gamma_n} v_2, \gamma_1 \dots \gamma_n \in L\}$

Theorem

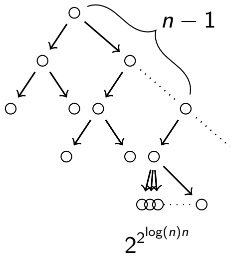
$\mathfrak{G}$ 2-CPG/ε, $g_0 \in \mathfrak{G}$;
L, K regular languages, $\xrightarrow{L}$ finitely branching
$\exists\ c, d$ s.t. $g_0 \xrightarrow{L} g_1 \xrightarrow{L} \dots \xrightarrow{L} g_n$ and $|\{g : g_n \xrightarrow{K} g\}| > 2^{2^{c+dn}}$
$\Rightarrow |\{g : g_n \xrightarrow{K} g\}| = \infty$.

Suffices: $\xrightarrow{L}$ finitely branching at $g_0, g_1, \dots, g_{n-1}$.

Application to trees

Example

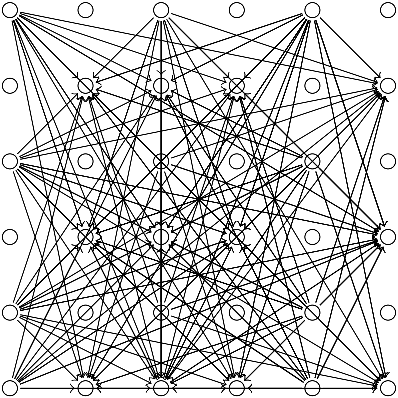
$\mathfrak{T} := (T, \mathrm{succ})$ with
$T := \{0\}^* \cup \{0^{n-1}1j : 0 \leq j \leq 2^{2^{\log(n)n}}\}$
is not a 2-CPG/ε



Proof.

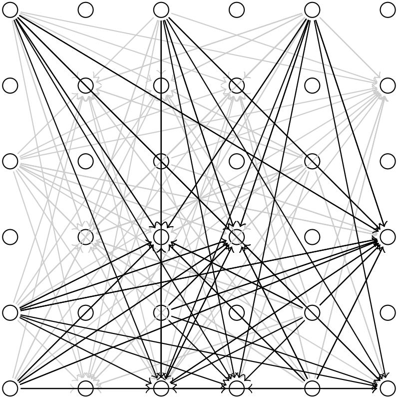
For $L = K = \Gamma$, we get c, d
Choose $n_0 > 2^{c+d}$ then $\log(n_0)n_0 > c + dn_0$
$\overset{\text{P.L.}}{\Rightarrow}$ $0^{n_0-1}1$ has infinitely many successors. $\square$

Application to Graphs



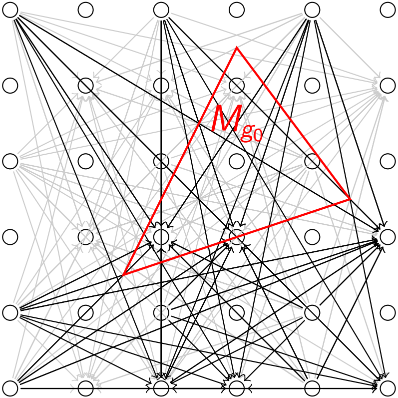
$\mathfrak{G}$: 2-CPG$/\varepsilon$

Application to Graphs



$\mathfrak{G}$: 2-CPG/ε
$\xrightarrow{L}$: L regular

Application to Graphs



$\mathfrak{G}$: 2-CPG$/\varepsilon$

$\xrightarrow{L}$: L regular

$(M_{g_0}, \xrightarrow{L}) \not\simeq \mathfrak{T}$ for

$M_{g_0} := \{g \in \mathfrak{G} : g_0 \xrightarrow{L^*} g\}$

Proof Strategy

1. Stacs'10:
   - Encoding of vertices of 2-CPG/ε in trees
   - $\xrightarrow{L}$ is represented by finite tree-automaton $\mathcal{A}_L$
2. Apply regular pumping lemma to $\mathcal{A}_L$.

- $\mathcal{T}$: set of all trees (finite binary Σ-labelled)
- automaton = (nondeterministic) finite tree-automaton

Definition

$R \subseteq \mathcal{T} \times \mathcal{T}$ binary relation of trees
$\mathcal{A}$ automaton
R automatic via $\mathcal{A}$: $\mathcal{A}$ accepts $t_1 \otimes t_2 \Leftrightarrow (t_1, t_2) \in R$ ($L(\mathcal{A}) = R$)

$$t_1 = \begin{tikzcd} & \text{a} \arrow[dl] \arrow[dr] & & \\ \text{b} & & \text{b} \arrow[dl] \arrow[dr] & \\ & \text{a} & & \text{b} \end{tikzcd} \qquad t_2 = \begin{tikzcd} & & \text{c} \arrow[dl] \arrow[dr] & \\ & \text{d} \arrow[dl] \arrow[dr] & & \text{c} \\ \text{d} & & \text{c} & \end{tikzcd}$$

$$t_1 \otimes t_2 = \begin{tikzcd} & & & a, c \arrow[dll] \arrow[drr] & & & \\ & b, d \arrow[dl] \arrow[dr] & & & & b, c \arrow[dl] \arrow[dr] & \\ \square, d & & \square, c & & a, \square & & b, \square \end{tikzcd}$$

Automatic Presentations

Definition

$R \subseteq M \times M$ relation over arbitrary set
$f : M \to \mathcal{T}$ injective function
$\mathcal{A}$ automaton
$(f, \mathcal{A})$ *automatic presentation of* R: $f(R)$ automatic via $\mathcal{A}$ where
$f(R) := \{(f(m_1), f(m_2)) : (m_1, m_2) \in R\}$

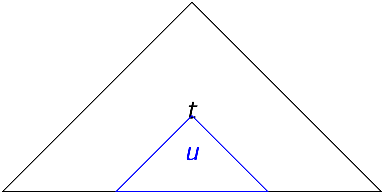
Theorem (Kartzow'10)

$\mathfrak{G}$ 2-CPG/ε with edge labels from Γ
$L \subseteq \Gamma^*$ regular,
then $\xrightarrow{L}$ has a tree-automatic presentation $(f, \mathcal{A})$.

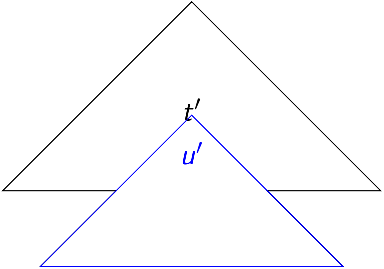
Regular Pumping for Tree-Automata

Lemma (regular pumping lemma)

$\mathcal{A}$ with d states
$\exists t \in L(\mathcal{A})$ *with* $|t| > d \Rightarrow |L(\mathcal{A})| = \infty$



$|u| \leq d$

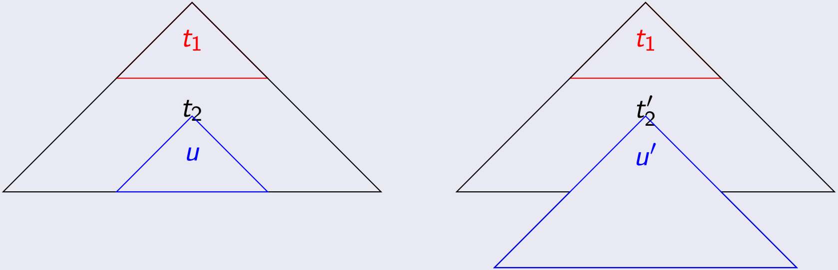


Pumping on Automatic Relations

Lemma

Automaton $\mathcal{A}$ with d states
R automatic via $\mathcal{A}$
Trees t_1, t_2 s.t. $|t_2| > |t_1| + d$
$(t_1, t_2) \in R \Rightarrow \{t : (t_1, t) \in R\}$ is infinite

Proof.



$\square$

Pumping on Automatic Relations

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Corollary

$|\{t : (t_1, t) \in R\}| > (|\Sigma| + 1)^{2^{|t_1|+d}} \Rightarrow \{t : (t_1, t) \in R\}$ *is infinite*

Proof.

There are $(|\Sigma| + 1)^{2^n}$ different Σ-labelled trees of depth n. □

The Pumping Lemma for 2-CPG

Lemma

Automaton $\mathcal{A}$ with d states
R automatic via $\mathcal{A}$
Trees t_1, t_2 s.t. $|t_2| > |t_1| + d$
$(t_1, t_2) \in R \Rightarrow \{t : (t_1, t) \in R\}$ *is infinite set.*

Lemma

$\mathfrak{G}$ *2-CPG/ε;*
L regular such that $\xrightarrow{L}$ is finitely branching.
There is a constant d such that $g_0 \xrightarrow{L} g_1 \xrightarrow{L} \dots \xrightarrow{L} g_n \Rightarrow |g_n| \leq |g_0| + dn$.

The Pumping Lemma for 2-CPG

Theorem

$\mathfrak{G}$ CPG/ε; L, K regular languages, $\xrightarrow{L}$ finitely branching
$\exists c, d$ *s.t.* $g_0 \xrightarrow{L} g_1 \xrightarrow{L} \dots \xrightarrow{L} g_n$ *and* $|\{g : g_n \xrightarrow{K} g\}| > 2^{2^{c+dn}} \Rightarrow$
$|\{g : g_n \xrightarrow{K} g\}| = \infty.$

Conclusion and Open Problems

Conclusion

- pumping lemma for 2-CPG/ε: tool for disproving membership
- regular reachability $\xrightarrow{L}$ on 2-CPG/ε is tree-automatic
- pumping lemma for finite automata applied to regular reachability yields pumping lemma for 2-CPG/ε.

Open questions

- pumping lemmas for higher level CPG
- pumping lemma for the whole hierarchy
- other techniques for disproving membership